Top Trading Cycles Mechanism

Qianfeng Tang (SUFE)

Summer School of Modern Economics
Nanchang University, July 2015
House allocation

Objects (houses) \( H = \{h_1, h_2, \ldots, h_n\} \) are to be allocated to agents \( N = \{1, \ldots, n\} \):

- Each object has exactly one copy
- Each agent \( i \) has a strict preference \( P_i \) over \( H \); \( P = (P_i)_{i \in N} \)
- A matching (allocation) is a bijection \( \mu : N \to H \)
- No monetary transfer allowed

Applications include kidney exchange, allocation of housing/offices, etc.
Matching criteria

A mechanism $\varphi$ selects a matching $\varphi(P)$ for each preference profile $P$; use $\varphi_i(P)$ to denote $i$’s assignment at $P$.

When allocating objects to agents, certain properties are desirable:

- A matching $\mu$ is **Pareto efficient** (for students) if there is no matching $\nu \neq \mu$ such that $\nu(i) R_i \mu(i), \forall i$ and $\nu(j) P_j \mu(j)$ for some agent $j$.
- $\varphi$ is **Pareto efficient** if it selects a Pareto efficient matching for all $P$.
- $\varphi$ is **strategy-proof** if no student can benefit from misreporting. Formally, if $\varphi_i(P_i, P_{-i}) R_i \varphi_i(P'_i, P_{-i})$, for all $i, P'_i, P_{-i}$.
- $\varphi$ is **group strategy-proof** if there do not exist $C \subseteq I$ and $P'_C$ such that for all $i \in C, \varphi_i(P_C, P_{-C}) R_i \varphi_i(P'_C, P_{-C})$, and for some $j \in C, \varphi_j(P_C, P_{-C}) P_j \varphi_j(P'_C, P_{-C})$. 
Serial dictatorship

Let’s begin with the simplest "good" mechanism

**Serial dictatorship**: An order, which is a bijection $f : N \rightarrow N$, is given *a priori*. The first agent (in the order) chooses her favorite object first, then the second agent chooses her favorite from what remain, and so on.

**Example**
Suppose $H = \{a, b, c\}$, $N = \{1, 2, 3\}$, and agents have the following preferences:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>c</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>a</td>
<td>c</td>
</tr>
</tbody>
</table>
Let $r$ be a relabeling (permutation) of houses and $P^r$ be the new preferences. A mechanism $\varphi$ is neutral (to the names of objects) if $\varphi(P^r) = r(\varphi(P))$.

DA with non-uniform priorities is not neutral, and DA with uniform priority is exactly SD with this priority as serial order.

SD is Pareto efficient, group strategy-proof, and neutral. Furthermore,

**Theorem (Svensson, 1999)**

A mechanism is group strategy-proof and neutral if and only if it is a serial dictatorship.

The necessity part is not difficult. See Svensson’s paper for proof of the sufficiency part.
A housing market is a profile \( \{(i, h_i)_{i \in N}, P\} \) such that:

- agent \( i \) owns house \( h_i \), \( \forall i \)
- agent \( i \) has strict preference \( P_i \) over \( H \)

Gale’s top trading cycles (TTC) mechanism:

Step 1 Each agent points to the owner of her most favorite house. Due to finiteness, there exists at least one cycle (including self-cycles) and cycles don’t intersect. Let agents in cycles trade and remove them.

Step \( k, k \geq 2 \) Repeat Step 1 with the remaining agents until all are removed.
Example

Suppose $N = \{1, 2, 3, 4, 5\}$ and each agent $i$ is endowed with house $h_i$. Consider preference profile

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_2$</td>
<td>$h_3$</td>
<td>$h_2$</td>
<td>$h_3$</td>
<td>$h_1$</td>
<td></td>
</tr>
<tr>
<td>$h_4$</td>
<td>...</td>
<td>...</td>
<td>$h_4$</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>$h_5$</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>
The core

A matching $\mu$ is in the core of the market if there exists no coalition $B \subseteq N$ and matching $\nu$ such that

- For any $i \in B$, $\nu(i)$ is owned by some agent in $B$
- For agents in $B$, matching $\nu$ Pareto dominates $\mu$

A core allocation is both individually rational (consider $B$ as singleton set) and Pareto efficient (consider $B$ as the whole set $N$)

**Theorem (Roth and Postlewaite, 1977)**

*The matching produced by Gale’s TTC is the unique core matching of housing market.*
Let $\mu$ be the TTC assignment. Suppose instead $\mu$ is not in the core

- By definition of core, $\mu$ is blocked by some coalition $B$ via matching $\nu$
- Let $i$ be the earliest matched agent in TTC among agents in $B$ who are strictly better off under $\nu$. Then $\nu(i)$ must have been removed earlier than $i$ in TTC
- Suppose $\nu(i) = h_j$, the house of agent $j$. Then $j \in B$ and is matched earlier than $i$ in TTC. By assumption on $i, j$ is not strictly better off under $\nu$, i.e., $\mu(j) = \nu(j)$
- Likewise, $j \in B$ implies that the owner of $\nu(j)$ is in $B$, is removed together with $j$ in TTC, and is matched the same under $\mu$ and $\nu$
- By induction, the agent who obtains $h_j$ in TTC is in $B$, and since $h_j$ is assigned to $i$ at $\nu$, this agent must be strictly improved at $\nu$. Since she is removed earlier than $i$, we have a contradiction
Proof: uniqueness

Suppose other than the TTC matching $\mu, \nu$ is another matching in the core

- Then for some $i, \mu(i) \neq \nu(i)$
- Note that first, $i$ cannot be any agent who is matched in the first round of TTC, because such agents obtain their most favorite at $\mu$ and $\nu(i) \neq \mu(i)$ implies that $\nu(i)$ is worse
- Therefore, $i$ can always block $\nu$ by forming a coalition with members in the same cycle in TTC
- By the same argument $i$ cannot be any agent in the second round of TTC, and so on
Ma’s characterization

A matching $\mu$ is **individually rational (IR)** if $\mu(i)R_ih_i$ for all $i$

**Theorem (Ma, 1994)**

A mechanism is strategy-proof, Pareto efficient and individually rational if and only if it is TTC.

That is, for housing markets, if a mechanism always selects the IR and Pareto efficient outcome, and is strategy-proof, then it must always select the core outcome.
Proof

Suppose a mechanism $\varphi$ satisfies all three axioms above. Let $P$ be any preference profile.

- First, consider any agent $i$ who trades in the first step of TTC. Suppose instead $\varphi_i(P) \neq \text{TTC}_i(P)$.
- If $i$ trades with herself in TTC, then since $\varphi$ satisfies IR, $\varphi_i(P) = \text{TTC}_i(P) = h_i$.
- Otherwise, $i$ trades with others in TTC. For simplicity, suppose it is a two-way cycle $i \leftrightarrow j$.
- Then $i$ top ranks $h_j$ and $\text{TTC}_i(P) = h_j$ is better than $\varphi_i(P)$.

Consider an alternative preference $P'_i : h_jh_i\emptyset$ of $i$.

- Since $\varphi$ is strategy-proof, $\varphi_i(P'_i, P_{-i}) \neq h_j$, because otherwise $i$ has incentive to misreport $P'_i$ when her true preference is $P_i$; hence due to IR, $\varphi_i(P'_i, P_{-i}) = h_i$.
- Consequently, $\varphi_j(P'_i, P_{-i}) \neq h_i$ and $j$ is worse off than in TTC.

Similarly, consider $P'_j : h_ih_j\emptyset$.

- Since $\varphi$ is strategy-proof, $\varphi_j(P'_i, P'_j, P_{-ij}) \neq h_i$ and due to IR, $\varphi_j(P'_i, P'_j, P_{-ij}) = h_j$.
- At the preference profile $(P'_i, P'_j, P_{-ij})$, $\varphi$ is not Pareto efficient, a contradiction. The rest of the proof follows from induction.
Application: Kidney exchange

Some background:

- National Organ Transplant Act, 1984, makes it illegal to buy or sell kidney
- over 90,000 patients are on waitlists for kidneys in the U.S. (with only around 17,000 donors in total)
- In 2006, there are 10659 transplants from deceased donors; 6428 transplants from live donors, and
- In a recent year, 4500 patients died while on the waiting list

- transplant from live donor survives much longer than from deceased donor
- issue of relative or friend donation: blood-type incompatibility or antibodies
- in 1986, the idea of live-donor paired kidney exchanges is proposed
- by 2005, only 10% of of live donor transplants are from live-donor exchanges
House allocation with existing tenants

Houses in $H_O$ are occupied by existing tenants and houses in $H_V$ are vacant. Application: Dormitory allocation and kidney exchange.

The **YRMH-IGYT** algorithm (Abdulkadiroglu and Sonmez, 1999) combines SD and TTC. Before it starts, a priority order $f$ of agents is exogenously given or randomly drawn.

Equivalently, it operates as follows:

**Step 1** All vacant houses point to the highest priority agent; every occupied house points to its owner. Trade and remove cycles.

**Step $k$, $k \geq 1$** Repeat Step 1 with the remaining houses and agents
General allocation rules

By now:

- For markets with ownership (e.g., housing market): Trading mechanisms
- For markets without ownership (e.g., house allocation): Serial dictatorship
- Mixed ownership (e.g., allocation with existing tenants): Mixed mechanism

For general allocation problems

- Endow: create a housing market; and then apply
- Trading mechanism
Hierarchical exchange rules

An **inheritance tree** of an object specifies

- to whom this object is initially endowed
- depend on the assignment history of previous owners, who inherits it

A **hierarchical exchange rule** (Papai, 2000) operates as follows:

Step 1 Objects are initially endowed to agents according to its inheritance tree. Agents trade according to TTC.

Step $k, k \geq 2$ After each previous trading history, after agents are removed, remaining objects are inherited by remaining agents according to the inheritance trees. Then trade and remove as usual.
Examples

**Sequential Dictatorship.** Let $N = \{1, 2, 3\}$ and $H = \{a, b, c\}$. Agent 1 chooses first. And if 1 chooses $a$ or $b$, then 2 chooses next; if 1 chooses $c$, then 3 chooses next.

**TTC in School Choice.** This mechanism is introduced by Abdulkadiroglu and Sonmez (2003); it extends Gale’s TTC from housing markets to priority-based allocation problems. In the algorithm, the priority list of a school is interpreted as the order that the students inherit seats from the school.

For each school choice problem, TTC operates as follows:

1. **Step 1** Each student points to her most favorite school; each school points to its highest priority student. There will be cycle(s). Match students in cycles with their favorite seats and remove them and those seats.

2. **Step $k$, $k \geq 2$** Repeat Step 1 with the remaining seats and students.
Unification of existing mechanisms

- **Serial dictatorship**: All objects are endowed to the first agent in the order and all left objects are always inherited by the next agent in the order. In trading, every agent points to herself.

- **TTC**: Each object is endowed to a different agent. No inheritance happen.

- **YRMH-IGYT**: All vacant houses are endowed and inherited according to the priority; each occupied house is owned by the respective existing tenant.

A mechanism is **reallocation-proof** if no pair of agents can gain from misreporting and swapping ex post in a self-enforcing way.

**Theorem (Papai, 2000)**

An allocation mechanism is group strategy-proof, Pareto efficient, and reallocation-proof if and only if it is a hierarchical exchange rule.