Two-sided Matching: Basics

Qianfeng Tang (SUFE)

Summer School of Modern Economics
Nanchang University, July 2015

School choice

In school choice, we assign seats at schools $S = \{s_1, \ldots, s_n\}$ to students $I = \{i_1, \ldots, i_m\}$:

- q_s : the **capacity** (quota) of school s
- $\succ_s$: the strict **priority structure** at school s; $i \succ_s j$ iff i has higher priority than j at s
- P_i : the strict **preference** of student i; $s' P_i s$ iff i prefers s' to s
- R_i : the extension of P_i. That is, $s' R_i s$ iff $s' P_i s$ or $s' = s$
- $\emptyset$: the **null school**, unlimited capacity
- $\succ \equiv (\succ_s)_{s \in S}, P \equiv (P_i)_{i \in I}$. And $\succ_{-s}, P_{-i}$ are defined as usual

A **school choice problem** is a pair $(P, \succ)$

Problems of interest

Desirable properties (criteria) for school choice mechanisms: Stability, Pareto efficiency, and strategy-proofness

Issues in market design:

- studying the trade-offs among these criteria
- design for weak priorities
- design for affirmative action

Summary of mechanisms

The popular mechanisms and their properties

	Stability	Efficiency	Stgy.Proofnes
DA	Y	N	Y
TTC	N	Y	Y
Boston	N	Y	N

Example: Inefficiency of DA

For a given $\succ$, the mechanism $DA^{\succ}(\cdot)$ is always optimally stable (for students), but may not be Pareto efficient

Example

In the problem $(P, \succ), S = \{s_1, s_2\}, q_{s_1} = q_{s_2} = 1$; $I = \{i, j, k\}$.

$\succ_{s_1}$	$\succ_{s_2}$	P_i	P_j	P_k
i	k	s_2	s_1	s_1
j	i	s_1	$\emptyset$	s_2
k				

s_1	s_2	$\emptyset$
$\boxed{j}, k$	$\boxed{i}$	
	$\boxed{k}, i$	
$\boxed{i}, j$		
		$\boxed{j}$
$\boxed{i}$	$\boxed{k}$	$\boxed{j}$

Group strategy-proofness

A mechanism φ is **group strategy-proof** if there do not exist $C \subset I$ and P'_C such that for all $i \in C, \varphi_i(P_C, P_{-C}) R_i \varphi_i(P'_C, P_{-C})$, and for some $j \in C, \varphi_j(P_C, P_{-C}) P_j \varphi_j(P'_C, P_{-C})$.

Definition

A mechanism φ is nonbossy if for all P, i, and $\tilde{P}_i, \varphi_i(P) = \varphi_i(\tilde{P}_i, P_{-i})$ implies $\varphi(P) = \varphi(\tilde{P}_i, P_{-i})$

Theorem

A mechanism is group strategy-proof if and only if it is strategy-proof and nonbossy.

DA is strategy-proof for students but not group strategy-proof, because it is bossy. In Example 1, j can benefit i and k by misreporting $P_j : \emptyset$, without affecting her own assignment.

Characterizing inefficient priority structures

For the priority structure given in Example 1, $DA^{\succ}(\cdot)$ is inefficient. Ergin (2002, Econometrica) provides a general characterization of such priority structures

Definition (Ergin, 2002)

Let $\succ$ be a priority structure and q be a vector of quotas (capacities). Then $(\succ, q)$ contains a cycle if there exist $s_1, s_2 \in S$, and $i, j, k \in I$, such that

1. $i \succ_{s_1} j \succ_{s_1} k \succ_{s_2} i$;
2. There exist (possibly empty) disjoint sets $I_1, I_2 \subset I \backslash \{i, j, k\}$ such that $I_1 \succ_{s_1} j$ and $I_2 \succ_{s_1} i, |I_1| = q_{s_1} - 1$, and $|I_2| = q_{s_2} - 1$.

A priority structure is **acyclical** if it does not contain any cycles. Note that a cycle is defined by both school priorities and school capacities.

Characterizing inefficient priority structures

Theorem (Ergin, 2002)

For any $(\succ, q)$, the following are equivalent:

1. $DA^{\succ}(\cdot)$ *is Pareto efficient;*
2. $DA^{\succ}(\cdot)$ *is group strategy-proof;*
3. $(\succ, q)$ *is acyclical.*

The proof of necessity is constructive and is quite invovled. We sketch the intuition for sufficiency

Proof intuition

Given an acyclic structure $(\succ, q)$, consider a preference profile in which all students in I_1 likes only s_1 and all students in I_2 likes only s_2

P_i	P_j	P_k	P_{I_1}	P_{I_2}
s_2	s_1	s_1	s_1	s_2
$\underline{s_1}$	$\underline{\emptyset}$	$\underline{s_2}$		

s_1	s_2	$\emptyset$
$\boxed{I_1, j}, k$	$\boxed{I_2, i}$	
	$\boxed{I_2, k}, i$	
$\boxed{I_1, i}, j$		
		$\boxed{j}$
$\boxed{I_1, i}$	$\boxed{I_2, k}$	$\boxed{j}$

Characterizing inefficient priority structures

Theorem (Ergin, 2002)

Let $r_s(i)$ be the rank of student i at school s. $(\succ, q)$ is cyclical iff there exist student i and school s_1, s_2 such that i's rank is larger than $q_{s_1} + q_{s_2}$ at s_1 or s_2, and $|r_{s_1}(i) - r_{s_2}(i)| > 1$.

Hence we see that acylicity does not impose conditions on the across-school priorities of upper class students.

Example

Acyclical priority structures: left table $q_{s_1} = 1, q_{s_2} = 2$; right table $q_{s_1} = 2, q_{s_2} = q_{s_3} = 3$.

$\succ_{s_1}$	$\succ_{s_2}$
i	**k**
j	**i**
k	

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
1	**2**	**5**
2	**3**	**4**
3	**5**	**1**
4	4	3
5	1	2

Related studies

Consistency. In Ergin's original paper, he also discusses about the consistency axiom, which we skip here

Robust stability. We see from Example 1 that if j misreports $P'_j : \emptyset$ where her true preference is P_j, then her DA assignment will be the same. But ex post, she can block the DA matching with s_1. Kojima (2011, TE) defines a mechanism to be robustly stable if no such manipulation is possible, and then characterizes robust stability mechanisms with acyclical structures

TTC and Kesten-cycles. For school choice problems, TTC and DA can be viewed as two competing mechanisms. Like Ergin (2002), who characterizes structures $(\succ, q)$ at which DA is efficient, Kesten (2006, JET) characterizes structures $(\succ, q)$ at which $TTC^{\succ}(\cdot)$ is a stable, or equivalently, at which $DA^{\succ}(\cdot)$ and $TTC^{\succ}(\cdot)$ are equivalent mechanisms.

When schools may have non-unit-capacities, even if at $(\succ, q)$ DA is Pareto efficient for all P (and also stable), TTC may generate different outcomes from DA. In other words, TTC is unnecessarily unstable

Example

Consider the left table in Example 7. We know that this $(\succ, q)$ is acyclical, hence $DA^{\succ}(P)$ is efficient for all P. For the preference in Example 1, DA matches s_1 to j and s_2 to i (this is efficient), but TTC matches s_1 to k.

Trade-offs among criteria

Suppose φ is a mechanism that Pareto dominates DA, then

- φ is not stable, because DA is already optimally stable
- φ is not strategy-proof, due to the following impossibility result (see e.g., Erdil, 2014, JET)

Theorem

If φ is a strategy-proof and nonwasteful mechanism, then there is no strategy-proof mechanism that Pareto dominates φ.

Proof intuition

Lemma

If ν weakly Pareto dominates a non-wasteful matching μ, then the same set of students is matched at ν and μ

Next, fix $\succ$. Suppose there exists ψ that dominates φ. Then there exists P, such that $\psi_i(P) R_i \varphi(P)$ for all i and $\psi_j(P) P_j \varphi_j(P)$ for some j. Let $\psi_j(P) = s$. Consider $P'_j : s\emptyset$.

$$
\begin{array}{llcl}
 & \varphi & & \psi \\
P_i & \varphi_i(P) & < & s \\
 & \Downarrow_{\text{SP of } \varphi} & & \uparrow_{\text{misreport}} \\
P'_i & \emptyset & \overset{Lemma}{\Longrightarrow} & \emptyset
\end{array}
$$

Example

Suppose ψ Pareto improves on DA by letting i and k trade in the Ergin cycle example. Then ψ is not strategy-proof

Proof. By definition, $\psi_k(P) = s_1$. At $P'_k : s_1\emptyset$, DA assigns $\emptyset$ to k and so wil ψ. Therefore, under ψ, k has incentive to report P_k when her true preference is P'_k

$\succ_{s_1}$	$\succ_{s_2}$	P_i	P_j	P'_k
i	k	s_2	s_1	s_1
j	i	s_1	$\emptyset$	$\emptyset$
k				s_2

s_1	s_2	$\emptyset$
$\boxed{j}, k$	$\boxed{i}$	
		$\boxed{k}$
$\boxed{j}$	$\boxed{i}$	$\boxed{k}$

Pareto improvement on DA

It has been empirically documented that the efficiency loss of DA can be significant in practice (Abdulkadiroglu, Pathak, and Roth, 2009)

We have also seen that Pareto improvement on DA hurts both stability and strategy-proofness

In general, mechanisms that improve on DA are expected to be difficult to manipulate, especially when the market is large.

Now, if we want to Pareto improve on DA, how should we do it?

Kesten (2010): School choice with consent

In the DA procedure of the Ergin cycle example, j is an "interrupter"–she is tentatively accepted by s_1, crowds out other students during the acceptance, but is later rejected by s_1

Observe that j can improve the assignments of k and i without hurting her own assignment, if she gives up her priority at s_1

Inspired by this, Kesten (2010) seeks to improve the efficiency of DA by obtaining students' consent to give up (some of) their priorities.

EADAM

Suppose the consent of some students have been obtained. Kesten's **efficiency-adjusted DA mechanism** (EADAM) operates as follows:

Round 0: Run DA for $(P, \succ)$

Round $k, k \geq 1$: Find the last consenting interrupter of round-$(k-1)$ DA, remove the interrupted school from her preference and re-run DA.
Stop when there are no more consenting interrupters

Kesten shows that EADAM is Pareto efficient when all students consent. Furthermore, under EADAM, whether a student consents or not, her assignment will not be affected. This important result makes sure that the students do not have incentive to not consent.

Example

Suppose $I = \{i_1, \dots, i_6\}, S = \{s_1, \dots, s_5\}$, and $q_s = 1$, except $q_{s_5} = 2$. Let $(P, \succ)$ be described by

Example

s_1	s_2	s_3	s_4	s_5
i_2	i_3	i_1	i_4	$\vdots$
i_1	i_6	i_6	i_3	
i_5	i_4	i_2	i_6	
i_6	i_1	i_3	$\vdots$	
i_4	$\vdots$	$\vdots$		
i_3				

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}
s_2	s_3	s_3	s_1	s_1	s_4
s_1	s_1	s_4	s_2	s_5	s_1
s_3	s_5	s_2	s_4	$\vdots$	s_3
$\vdots$	$\vdots$	$\vdots$			s_2
					s_5

The DA procedure:

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_5}, i_4$	$\boxed{i_1}$	$\boxed{i_2}, i_3$	$\boxed{i_6}$	
2	$\vdots$	$i_1, \boxed{i_4}$	$\vdots$	$i_6, \boxed{i_3}$	
3	$i_5, i_6, \boxed{i_1}$	$\vdots$		$\vdots$	
4	$\vdots$		$i_2, \boxed{i_6}$		$\boxed{i_5}$
5	$\boxed{i_2}, i_1$		$\vdots$		$\vdots$
6	$\vdots$		$\boxed{i_1}, i_6$		
7		$\boxed{i_6}, i_4$	$\vdots$		
8		$\vdots$		$\boxed{i_4}, i_3$	
9		$i_6, \boxed{i_3}$		$\vdots$	
10	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_1}$	$\boxed{i_4}$	$\boxed{i_5}, \boxed{i_6}$

EADAM procedure:

Round 1 the last interrupter-school pair of the DA procedure is (i_6, s_2). Remove s_2 from P_{i_6} and re-run DA

Round 2 the last interrupter-school pair of round-1 DA is (i_6, s_3). Remove s_3 from P_{i_6} and re-run DA

Round 3 the last interrupter-school pair of round-2 DA is (i_5, s_1). Remove and re-run...

Round 4 the last interrupter-school pair of round-3 DA is (i_6, s_1). Remove and re-run...

Round 5 no more interrupters. Stop

Tang and Yu (2010): A new perspective on EADAM

Tang and Yu revisit Kesten's school choice with consent idea and provides a more intuitive and accessible approach. Incentives for consent are important in the design

When will a student consent to give up their own hope to help others, while knowing that everybody may be helped? The simple answer is: when a student find herself cannot be Pareto improved anymore

Now, which students are Pareto unimprovable?

Underdemanded schools

Definition

At matching μ, school s is an **underdemanded school** if no student prefers s to her own assignment

Observations:

- a school s is underdemanded at $DA(P, \succ)$ if and only if it never rejected any student during the DA procedure;
- since DA is non-wasteful, any Pareto improvement upon the DA matching must be through trading cycles

Lemma

If matching μ Pareto dominates $DA(P, \succ)$, and student i is matched with an underdemanded school at $DA(P, \succ)$, then $\mu(i) = DA(P, \succ)(i)$

We can further generalize this concept to **essentially underdemanded** schools

Underdemanded schools

The **simplified EADAM** we propose runs as follows:

Round 0 Run DA for the problem $(P, \succ)$.

Round $k, k \geq 1$

1. Consider last round's DA, settle the matching at underdemanded schools and remove them with the admitted students.
2. For each non-consenting student i who was removed, if i desires a remaining school s, remove all students with lower priority than i from $\succ_s$.
3. After that, re-run DA for the remaining schools and students

Stop when all schools are removed.

Example revisited

The simplified EADAM is well-defined, and since at least one school is removed in each round, the algorithm stops within $|S \cup \{\emptyset\}| = m + 1$ rounds

Let's revisit Kesten's Example. Suppose all students consent. At the end of $DA(P, \succ)$, s_5 is underdemanded, and is matched with i_5, i_6.

Round 1 Remove s_5 with i_5, i_6. Re-run DA, we have

	s_1	s_2	s_3	s_4
Step 1	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_2}, i_3$	
Step 2				$\boxed{i_3}$
Step 3	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_2}$	$\boxed{i_3}$

Round 2 s_1, s_2, s_4 are underdemanded, remove. The process "essentially" ends

Properties

Theorem

The simplified EADAM is Pareto efficient when all students consent and constrained efficient otherwise.

Since in the algorithm, a student's consent will be used only if her assignment cannot be further improved, the following becomes transparent. And such transparency is very important for a mechanism to be put into practice

Theorem

The assignment of any student does not change whether she consents or not.

Unification and equivalence

Now, we can unify our approach of focusing on underdemanded schools and Kesten's approach of focusing on lastly rejected interrupters

Lemma

If student i is the last interrupter rejected in the DA procedure, then $DA(P, \succ)(i)$ is essentially underdemanded and hence i is Pareto unimprovable

That is, Kesten's approach is simply another implementation of focusing on unimprovable students. As a result, the simplified EADAM and Kesten's EADAM are outcome equivalent

Weak priorities (Erdil and Ergin, 2008)

In practice, schools often rank students into tiers and hence have weak priorities over students. Consider any school choice problem $(P, \succeq)$, where $\succeq$ is a weak priority structure

Since the DA procedure is defined only for strict priorities and preferences, before running DA, we need to break ties in the weak priority lists. A tie-breaking rule τ for $\succeq$ is an exogenous permutation of the set of students I

Given a problem $(P, \succeq)$, matching μ is constrained efficient if it is stable and is not Pareto dominated by any other stable matching. However, tie-breakers do not necessarily bring us the constrained-efficient matching

Example

Consider a simple variation of the Ergin cycle Example. Now, j and k have the same priority at s_1

$\succeq_{s_1}$	$\succeq_{s_2}$
i	k
j, k	i

P_i	P_j	P_k
$\underline{s_2}$	s_1	$\underline{s_1}$
$\boxed{s_1}$	$\underline{\emptyset}$	$\boxed{s_2}$

The two stable matchings are marked out by underlines and boxes. The tie-breaking rule either breaks $\succeq_{s_1}$ as $i \succ k \succ j$ or as $i \succ j \succ k$, and the corresponding DAs produces the two stable matchings, respectively

$DA^{\tau}(\cdot, \succeq)$ is not constrained efficient for the tie-breaker $i \succ j \succ k$. Yet still, due to the impossibility result, we cannot improve its efficiency without sacrificing strategy-proofness

Stable improvement cycles

Fix a problem $(P, \succeq)$ and a matching μ. For each school x, let D_x be the highest $\succeq_x$-priority students among those who desire x (i.e., who prefer x to their assignmentes at μ).

Definition

A **stable improvement cycle** (SIC) consists of distinct students $i_1, \ldots, i_n \equiv i_0$ ($n \geq 2$) such that for each $l = 0, \ldots, n-1$,

(i) i_l is matched to some school at μ;

(ii) i_l desires $\mu(i_{l+1})$; and

(iii) $i_l \in D_{\mu}(i_{l+1})$

Given μ and a SIC at μ, if we let students in this SIC trade seats, the newly obtained matching will still be stable w.r.t. $(P, \succeq)$

SIC algorithm

Theorem

Fix $(P, \succeq)$ and a stable matching μ. If μ is not constraint efficient, then it admits a SIC

Therefore, starting with an arbitrary stable matching, we can achieve constrained efficiency by iteratively find and implement SICs. Erdil and Ergin propose the following SIC algorithm:

Step 0. Select a tie-breaking rule τ and run DA to obtain $DA^{\tau}(P, \succeq)$.

Step $t \geq 1$. At $\mu^0 \equiv DA^{\tau}(P, \succeq)$, find an SIC: for schools x and y, let $x \rightarrow y$ if some student i matched at x who desires y, and $i \in D_x$. If there are cycles, *select* one and implement the trading. Denote the newly obtained matching by μ^1, and iterate the process. If no cycle, stop.

For the previous example, at the DA outcome under the tie-breaker $i \succ j \succ k$, the SIC algorithm trades the cycle $s_1 \leftrightarrow s_2$

The SIC algorithm is similar to but different from TTC:

- The cycles here are stable improvement cycles; students are pointing to all schools that are better than their current match. While in TTC, each agent points to her most favorite school
- For convenience, the algorithm is described through the pointings among schools instead of that among students. Each school may point to none or multiple other schools. Hence, each school may be involved in multiple cycles, and cycle-selection is an issue (the simple way is to randomly pick one).

Simplified EADAM for weak priorities

The simplified EADAM (Tang and Yu, 2010) can also be used to simplify the process of handling weak priorities, by assuming that each student consents to give up her priorities after tie-breaking to students with the same original priority

At the DA matching, we can iteratively match and remove students matched at underdemanded schools, for each student i removed and each remaining school s that i desires, we remove all students with strictly lower s-priority than i from $\succeq_s$

Revisit the example above

Since the simplified EADAM recovers constrained efficiency within $|S \cup \{\emptyset\}| = m + 1$ rounds, computationally it is very fast. Furthermore, it does not suffer from the issue of cycle-selection and hence is more tractable.

Affirmative action

Let I be the set of students, I^M be the set of majority students, and I^m be the set of minority students

Stability can be strengthened accordingly when the following affirmative action policies are taken into consideration:

- Majority quota: the number of majority students matched to school s cannot exceed the majority quota q_s^M
- Minority reserve: if the number of minority students matched to school s is less than the minority reserve r_s^m, then minority students are always preferred to majority students.

Example

The following example illustrates that the spirit of affirmative action cannot be respected by setting majority quotas

Example (Kojima, 2012, GEB)

Suppose $S = \{s_1, s_2\}, I = \{i, j, k\}, I^M = \{i, j\}$, and $I^m = \{k\}$. And $q_{s_1} = 2, q_{s_2} = 1$. The preferences are:

$\succ_{s_1}$	$\succ_{s_2}$
i	j
j	k
k	i

P_i	P_j	P_k
s_1	s_1	s_2
$\emptyset$	s_2	s_1

Let $q^M_{s_2} = 1$. Both when $q^M_{s_1} = 2$ and $q^m_{s_1} = 1$, there is only one stable matching, respectively:

s_1	s_2
i, j	k

s_1	s_2
i, k	j

So we see that under Majority quota, when more stringent quota is imposed on majority students, the minority students may become worse off

This is not the case for Minority reserve

The DA algorithm modified for minority reserves (Hafalir et. al, 2013, TE):
Runs as usual as the student-proposing DA, except that when facing applications at any step, each school always accepts the top minority applicants first, up to the reserve q_s^m, and then consider the rest of the applicants by priority